# Chapter 3 Fourier Series

discussion in setting of interval $[-L, L]$
(similar for interval $[0, L]$, later)

**Def.** Given a function $f(x)$ on $[-L, L]$
its Fourier Series is given by

$$\boxed{a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x\,dx, \quad b_n = \frac{1}{L} \int \ldots \sin \ldots dx$$

# Basic Questions

(1) Does Fourier series always converge?

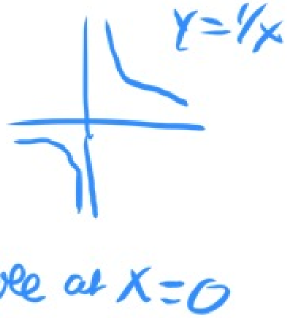
(2) " " " at $x$ coincide with $f(x)$?

(3) For which functions does (1) and (2) hold?

## Some Definitions

Let $f: [a,b] \to \mathbb{R}$

(a) $f$ continuous if graph of $f$ has <u>no</u> jumps and <u>no</u> poles (precise def. see Math 140 or 142)


or

(b) $f$ is piecewise smooth if

- $f$ differentiable for all but finitely many points in $[a,b]$
- $f'$ is continuous except for finitely many jumps

no poles allowed!

Examples:

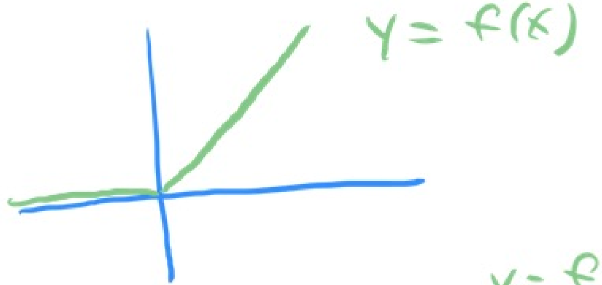
(a) $f(x) = x^2$

smooth $\Rightarrow$ piecewise smooth

($f'(x) = 2x$ is continuous everywhere)

(b)

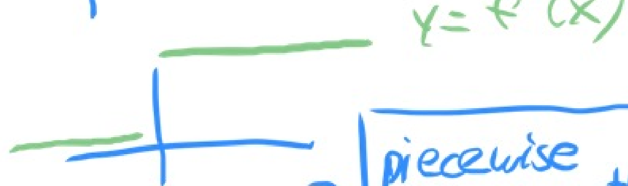
$$f(x) = \begin{cases} 0 & x \leq 0 \\ x & x > 0 \end{cases}$$




f continuous

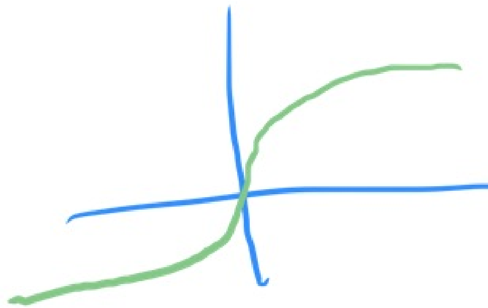
$$f'(x) = \begin{cases} 0 & x < 0 \\ 1 & x > 0 \end{cases}$$

not differentiable at 0




only one jump $\Rightarrow$ $\boxed{\text{piecewise smooth}}$

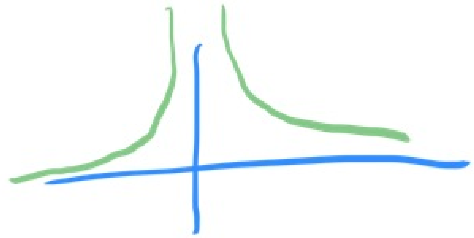
(c) $f(x) = x^{1/3}$




$f$ continuous ✓

$f'(x) = \frac{1}{3} x^{-2/3}$

$= \frac{1}{3x^{2/3}}$

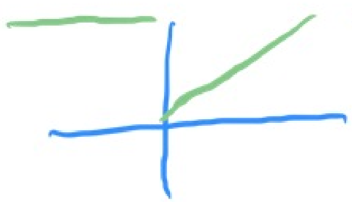


$f'$ has pole at $x=0$

$\Rightarrow$ NOT piecewise smooth

**Def.** $f(x_+) = \lim_{\substack{y \to x \\ y > x}} f(y)$, $f(x_-) = \lim_{\substack{y \to x \\ y < x}} f(y)$

**Ex.** $f(x) = \begin{cases} 1 & x < 0 \\ x & x \geq 0 \end{cases}$




$f(0+) = \lim_{\substack{y \to 0 \\ y > 0}} f(y) = \lim_{y \to 0} y = \boxed{0}$

$f(0-) = \lim_{\substack{y \to 0 \\ y < 0}} f(y) = \lim_{y \to 0} 1 = \boxed{1}$

# Fourier's Theorem

Let $f: [-L, L] \to \mathbb{R}$ be a piecewise smooth function and let

$$a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi}{L} x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{L} x$$

be its Fourier series.

Then the Fourier series

- converges to $f(x)$ if $f$ continuous at $x$
- converges to $\frac{1}{2}[f(x_+) + f(x_-)]$ if $f$ has a jump at $x$
- converges to $\frac{1}{2}[f(L-) + f(L_+)]$ for $x = L$ or $x = -L$

here:

$$f(x_+) = \lim_{\substack{y \to x \\ y > x}} f(y)$$

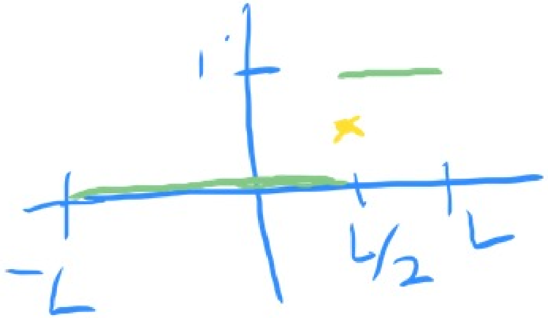
$$f(x_-) = \lim_{\substack{y \to x \\ y < x}} f(y)$$

Example:


$f(0_+) = 1$

$f(0_-) = 0$

Example: let $f(x) = \begin{cases} 1 & L/2 \leq x < L \\ 0 & -L \leq x < L/2 \end{cases}$




$$\Rightarrow \quad f(L/2+) = \lim_{\substack{y \to L/2 \\ y > L/2}} f(y) = \lim_{y \to L/2} 1 = 1$$

$$f(L/2-) = \lim_{\substack{y \to L/2 \\ y < L/2}} f(y) = \lim 0 = 0$$

$\Rightarrow$ value of Fourier transform at $L/2$

$$= \frac{1}{2}\left(f(L/2+) + f(L/2-)\right) = \frac{1}{2}(1+0) = \frac{1}{2}$$

Similarly, we calculate

$$f(L-) = \lim_{\substack{y \to L \\ y < L}} f(y) = 1$$

$$f(-L_+) = \lim_{\substack{y \to -L \\ y > L}} f(y) = 0$$

$\Rightarrow$ value of Fourier transform at $L = \frac{1}{2}(1+0)$

$$= \frac{1}{2}$$

$$= \text{value at } -L$$

**Def.** Periodic extension of $f : [-L, L] \to \mathbb{R}$ is a function defined on all of $\mathbb{R}$ by shifting the graph of $f$ by multiples of $2L$

Example: $L = 1 \qquad f(x) = x$